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One-side riddled basin below and beyond the blowout bifurcation

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In this Rapid Comunication we report a phenomenon of a *one-side* riddled basin where one side of the basin of attraction of an attractor on an invariant subspace (ISS) is globally riddled, while the other side is only locally riddled. This kind of basin appears due to the symmetry breaking with respect to the ISS. This one-side riddled basin can even persist *beyond* the blowout bifurcation, contrary to the previously reported riddled basins which exist only *below* the blowout transition. An experimental situation where this phenomenon can be expected is proposed.

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Recently, systems with an invariant subspace (ISS) attracted a lot of research interest [1–10]. The existence of an ISS typically requires that the system has some kind of symmetry. A situation where a symmetry can appear naturally is the synchronization of coupled identical chaotic units. The synchronization phenomena have been extensively studied in the context of laser dynamics, electronic circuits, chemical and biological systems, and secure communication. Complete, generalized, phase, and lag synchronizations of chaotic oscillators have been described theoretically and observed experimentally [1–12].

For systems possessing an ISS, a central question is the transverse stability of the ISS, i.e., whether an orbit starting from the vicinity of ISS will finally approach it or diverge from it. A quantity measuring this stability is the transverse Lyapunov exponent (TLE), which measures the growth rate of transverse perturbations. If the TLE is negative, a transverse perturbation will decrease gradually and the ISS is transversely stable. Otherwise, it is unstable. The problem turns out to be quite complex when there is a chaotic attractor on the ISS, since the value of TLE depends in a fundmental way on the choice of an invariant measure on the attractor [13]. It is known that invariant measures are not unique for a chaotic attractor due to the fact that there are uncountably many unstable periodic orbits (UPOs) embedded in it [14]. Corresponding to each UPO, there is an invariant measure whose support is on this orbit. To each invariant measure corresponds a TLE. In general, all of these TLEs have different values. With a change of some (normal) parameter [5], e.g., the coupling strength for the coupled systems, these TLEs will change their signs at different values of the parameter. The point where the *maximal* TLE crosses zero is called a riddling bifurcation [6]. Below this bifurcation, the attractor on ISS is asymptotically stable and the coupled systems are strongly synchronized. After this bifurcation, it is only an attractor in the sense of Milnor [2,15], i.e., a dense set of points escape from its vicinity. Its basin of attraction can be locally or globally riddled depending on the global dynamic of the system [4,5,9]. We have the weak synchronization in this region. The parameter value where $\Lambda_\perp$, which corresponds to the natural measure of the whole chaotic attractor, is equal to zero is called a blowout bifurcation [4]. Beyond it, the whole chaotic attractor on the ISS becomes unstable in average and the coupled chaotic units become desynchronized.

A UPO on an ISS can lose its transverse stability through various local bifurcations. Possible candidates are pitch-fork [6], period-doubling [8,9], transcritical, or Hopf bifurcations [10]. All of these cases have been reported except the transcritical one [11]. In the previous studies of the riddled basin, the systems used are of much stronger symmetry than is necessary for the existence of an ISS. For example, in the case of chaos synchronization, symmetric coupling is usually used while using of identical chaotic units is already enough to produce an ISS, independently on the coupling method. In this Rapid Communication, we address the situation when the unnecessary symmetry is released. To illustrate our finding, we use a simple two-dimensional map which possesses an ISS while the whole system is asymmetric with respect to this ISS. Due to this asymmetry, the periodic saddles embedded in a chaotic attractor in the ISS lose their transverse stability through transcritical bifurcation. We found that the basin of attraction of the chaotic attractor on the ISS can be globally riddled from one face while only locally riddled from another face. This type of riddled basin can even be maintained *after* the blowout bifurcation.

We consider the following general class of dynamical systems popularly used in the literatures [6,7]:

$$\begin{aligned} x_{n+1} &= f(x_n) + (\text{high-order terms in } y_n), \\ y_{n+1} &= g(x_n, p) y_n + (\text{high-order terms in } y_n). \end{aligned} \tag{1}$$

In the former works [6,7], only odd powers of y_n were present on the right-hand side of the second equation in Eq. (1). In that case, the system has the symmetry $y \to -y$. The symmetry $y \to -y$ is appropriate for the coupled chaotic units when the coupling is symmetric [8,9]. Obviously, this symmetry is not necessary for the presence of the ISS $y=0$. So, we would like to include the even powers of y_n and release this symmetry. For simplicity and easiness of illustration, a two-dimensional version of Eq. (1) is used,

$$\begin{aligned} x_{n+1} &= 2x (\text{mod } 1) + q y_n^2, \\ y_{n+1} &= (p x_n + a) y_n (1 + b y_n - c y_n^2), \end{aligned} \tag{2}$$

where p, q, a, b, and c are positive real constants.

For any choice of the control parameters, we have $y_n = 0$ for $n > 1$ if $y_0 = 0$, i.e., $y_n = 0$ is an ISS of the whole

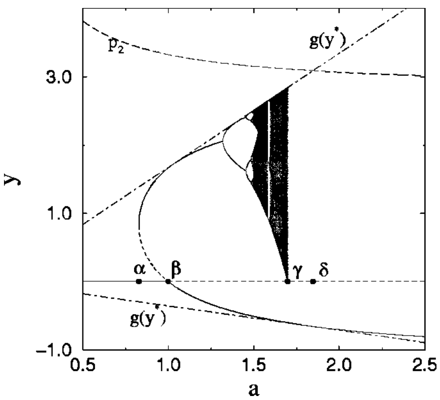

FIG. 1. Bifurcation diagram for the case of $p=0$: $y_{n+1}=g(y_n)=ay_n(1+by_n-cy_n^2)$.

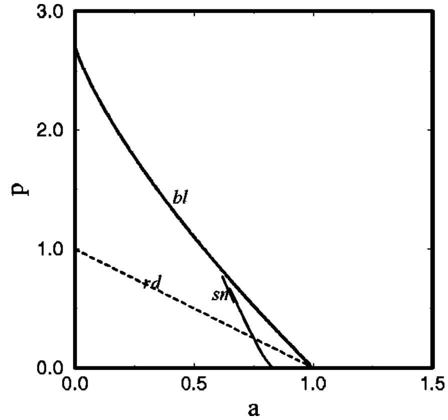

FIG. 2. Phase diagram for the two-dimsensional system. The curves bl, rd, and sn are for the blowout, riddling, and saddle-node-like bifurcations, respectively.

system. On this ISS, the system possesses a chaotic attractor from the Bernoulli shift map $f(x)=2x(\text{mod }1)$. Throughout this paper, the values $q=0.001$, $b=0.5$, and $c=0.3$ are kept fixed and a and p are used as control parameters..

Let us first briefly review the dynamics in the case $p=0$ (Fig. 1), where the dynamics of the variable y_n becomes uncoupled from that of x_n. With increasing of the parameter a, the fixed point $y_n=0$ loses its stability through a transcritical bifurcation at $a_\beta=1.0$. In the region $y<0$, another stable fixed point branches off smoothly from $y=0$ at the transcritical bifurcation. This is the soft branch. In the region $y>0$, a pair of fixed points appears simultaneously at $a_\alpha=0.83$, i.e., prior to the loss of the stability of $y=0$ at $a=1.0$. This is the hard branch. In the parameter interval $[a_\alpha, a_\beta]$, the system has two coexisting attractors. One can observe hysteresis when changing the parameter a in different directions. At $a_\gamma=1.697$, the second iteration of the critical point [16] $y^*=1.747$ on the hard branch, which is one of the maximum points of $g(y)=ay(1+by-cy^2)$, collides with $y=0$ which is on the boundary separating the basins of attraction of two attractors in the regions $y>0$ and $y<0$. Beyond this point, all the trajectories starting from the region $y>0$ will finally enter the region $y<0$; the attractor on the hard branch ceases to exist. At $a_\delta=1.847$, the first iteration of the critical point y^* collides with the unstable period-2 orbit which is on the boundary between the finite attractor and the attractor at infinity. After that, trajectories starting from the region $y>0$ will escape eventually to infinity. Similar boundary crises for the soft branch are at $a_\epsilon=3.29$ and $a_\zeta=4.26$; these points are outside the scope of our plot. Obviously, the two branches of the transcritical bifurcation are asymmetrical with respect to $y=0$. It is just this asymmetry that leads to the one-side riddled basin described below. This low level of symmetry (or say degeneracy) also implies that the one-side riddled basin is expected to be observed more commonly in contrast to other cases of riddled basin.

The TLE of the chaotic attractor on the ISS is given by $\Lambda_\perp=\int_0^1\ln|px+a|\rho(x)dx=\int_0^1\ln|px+a|dx=1/p(p\ln p-a\ln a-p+a)$. The blowout bifurcation occurs as $\Lambda_\perp=0$, which defines the curve $p\ln p-a\ln a-p+a=0$ in the parameter plane (p,a) (see Fig. 2). When changing the parameters p and a, four different scenarios of blowout bifurcations can be found in this system. The attractors and their basins before and after the bifurcations are listed in Table I.

To give an example of the first scenario, let us fix the parameter $a=0$ and increase p. For $p<1$, all the UPOs embedded in the chaotic attractor on ISS are stable in the transverse direction. The chaotic attractor is asymptotically stable in this case. As p is increased larger than $a_\beta=1.0$, the fixed point $x_n=1$ in the ISS becomes unstable in the transverse direction and a series of tonguelike structures open at the fixed point and its preimage. Points in these tongues can escape from the vicinity of the ISS. But they will finally return there due to the global nonlinearity of the system. The basin of attraction of the chaotic attractor on the ISS is only locally riddled now. Till the moment $p=a_\delta$, where the boundary of the basin of attraction and the attracting area collides at a period-2 orbit, the points escaping from the

TABLE I. Four scenarios of blowout bifurcations.

Scen.	*Example of Para.*	*Attra.* $\Lambda_\perp<0$	*Basins* $\Lambda_\perp<0$	*Attra.* $\Lambda_\perp>0$	*Basins* $\Lambda_\perp>0$
1	$a=0$ $p_c=2.718$	$y=0$	glob. ridd. by $+\infty$	$y=-c'$	glob. ridd. by $+\infty$
2	$a=0.4$ $p_c-1.357$	$y=0$	loc. ridd.	$y=-c'$	$y>0$ & $y<0$
3	$a=0.5$ $p_c=1.103$	$y=0$	loc. ridd.	$y=+c$ $y=-c'$	$y>0$ $y<0$
4	$a=0.7$ $p_c=0.635$	$y=0$ $y=+c$	glob. ridd. by +c	$y=+c$ $y=-c'$	$y>0$ $y<0$

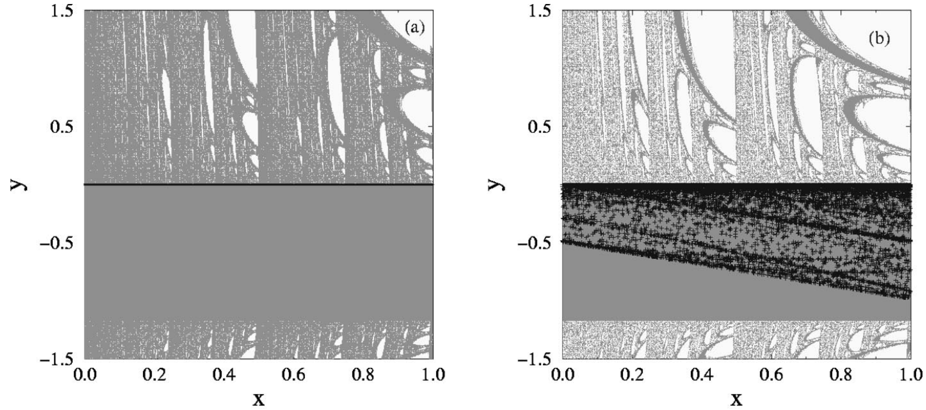

FIG. 3. One-side riddled basin for the case $a=0.0$, (a) $p=2.3$ before and (b) $p=2.75$ after the blowout bifurcation. The attractor is marked by black + and its basin of attraction is marked by gray dots.

vicinity of the ISS can go to another attractor at infinity and the basin of attraction for the chaotic attractor on the ISS is globally riddled in this case [see Fig. 3(a)]. It is important to keep in mind that the collision only happens on the half plane $y_n>0$. The points on the negative half plane $y<0$ still have no possibility to go to another attractor except the one on the ISS. So, the basin of attraction of the chaotic attractor on the ISS is globally riddled from one side $(y>0)$ and locally riddled from another side $(y_n<0)$. We call it a *one-side* riddled basin. With further increase of *p*, the chaotic attractor on the ISS becomes transversely unstable at $p=e=2.718\,28\ldots$. This is the blowout bifurcation. Beyond this bifurcation, the chaotic attractor expands its size in the $y<0$ direction, while the ISS keeps to be a part of its boundary. Since the ISS becomes transversely unstable, almost all points in its vicinity will locally escape. Due to the crisis at $p=a_\gamma$, a part of points in the half plane $y>0$ can be mapped to the negative plane $y<0$ after their escaping from the vicinity of the ISS. Now, in the positive half plane, there is a dense set of points which escape to infinity, and there is also a dense set of points which go to the attractor in the negative half plane. Both of these sets contact the ISS, which is part of the contour of the attractor in the negative plane. So, we say that its basin is globally riddled by that of the attractor at infinity. Or, in another words, the one-side globally riddled basin is maintained *beyond* the blowout bifurcation [see Fig. 3(b)]. This globally riddled basin exists till the crisis at $p=a_\delta$, where the attractor in the half plane $y<0$ in the vicinity of the ISS ceases to exist [17]. So, here the globally riddled basin exists in a large parameter region *beyond* the blowout bifurcation. We emphasize that the riddled basin here has some difference as compared to the conventional case: First, it is in the desynchronization region beyond the blowout bifurcation, while the riddled basin is conventionally related to the weak synchronization region below the blowout bifurcation. Second, the dynamical behaviors are different. Consider a small vicinity of the ISS. For conventional riddling, a set of finite measure in this region will escape and the measure of this set approaches zero as the size of vicinity decreases to zero. In our case, almost all points will escape and the measure of the escaping points is 1 even when the size of the vicinity approaches zero. However, due to the global nonlinearity, a part of them will be mapped to the negative half plane and finally evolves to the attractor there. This scenario of the blowout bifurcation can be observed for $0<a<0.31$.

For $0.62<a<1.0$, another scenario of the blowout bifurcation can be observed. As an example, we fix the parameter $a=0.7$ and increase the value of *p* (see Fig. 4). The basin of attraction of the attractor on the ISS becomes locally riddled as *p* crosses the line $a+p=1.0$ for the riddling bifurcation. At $p=0.458$, a chaotic attractor and a chaotic saddle appear simultaneously through a *saddle-node-like* bifurcation [18] in the half plane $y>0$. After that, the basin of the attractor on the ISS is globally riddled by that of the newly appeared chaotic attractor on the half plane $y>0$, while it is only locally riddled on the other face of the ISS. Here we have again the one-side riddled basin. Beyond the blowout bifurcation at $a=0.63$, the chaotic attractor on the ISS becomes transversely unstable and it expands its size in the half plane $y<0$. All points in the half plane $y<0$ go to this new attractor, while those in the half plane $y>0$ go to the attractor that appeared through the saddle-node-like bifurcation. For other values of parameters *p* and *a*, the other two scenarios of the blowout bifurcation can be observed. Details about them will be given elsewhere.

Now we propose an experiment where this one-side riddling can be observed. The experimental system is a gravitationally buckled, amorphous magnetoelastic ribbon driven parametrically by a time-varying magnetic field (see Fig. 5). It has been used in the experimental study of chaos control,

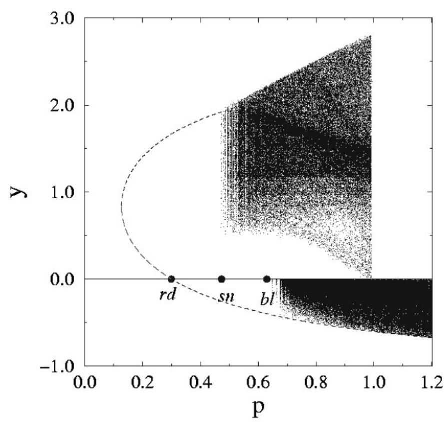

FIG. 4. Bifurcation diagram of the case of $a=0.7$. That of the fixed point $x_n=1$ is also plotted to guide eyes.

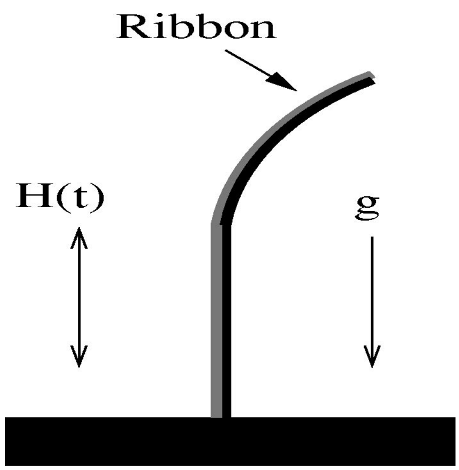

FIG. 5. Experimental configuration. The ribbon consists of two layers of different magnetoelastic properties. The vertical field $H(t)$ is modulated chaotically.

trajectory targeting, crisis induced intermittency, noise-induced crisis, strange nonchaotic attractor, and stochastic resonance [19]. The critical height, h_c, at which a vertical column will buckle depends on its stiffness. By changing the value of the vertical field H, the Young modulus, $E(H)$, of the ribbon can be altered, thereby setting the value of h_c alternately greater than and less than the actual height of the ribbon. Here we suggest to use a chaotically varying vertical field $H(t)$ to modulate the stiffness of the ribbon. So, the nonbuckled state with $y=0$, which is an ISS of the studied system, contains a chaotic attractor. Also, a two-layer ribbon with the two layers of the different stiffness or magnetoelastic properties should be used instead of the original symmetric one. This replacement will not destroy the nonbuckled state (the ISS), but will introduce some asymmetry in this system. Due to this asymmetry, it is expected that the loss of transverse stability of the nonbuckled state on the ISS corresponds to the transcritical bifurcation. Thus the one-side riddled basin reported in this paper can be observed in this proposed experiment.

In conclusion, we would like to outline the principal results of this paper. We demonstrated that in a system with a lower level of symmetry compared to other systems formerly used in the studies of riddled basins, a different type of riddled basin (the *one-side* riddled basin), is observed where the attractor on the ISS is globally riddled from one face while it is locally riddled from another face. This one-side riddled basin can even persist *beyond* the blowout bifurcation. This gives the first example where a globally riddled basin can be observed after the blowout bifurcation for an attractor off the ISS. Due to the low level of symmetry needed, it is expected that this different type of riddled basin can be observed more commonly. We also proposed an experiment where this phenomenon is expected to be observed.

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